Discretization of multidimensional submanifolds associated with Spin-valued spectral problems

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Abstract

We present a large family of $\mathrm{Spin}(p,q)$-valued discrete spectral problems. The associated discrete nets generated by the so called Sym-Tafel formula are circular nets (i.e., all elementary quadrilaterals are inscribed into circles). These nets are discrete analogues of smooth multidimensional immersions in $\mathbf{R}^m$ including isothermic surfaces, Guichard nets, and some other families of orthogonal nets.

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One of the most important topics in the classical differential geometry was to study special coordinates (nets) on surfaces (and submanifolds) and various transformations, associated with the names of Bianchi, Bäcklund, Darboux, Ribaucour, Levy, Combescure, Jonas and others [24, 25]. Recently, one can observe a rapid development of a discrete analogue of differential geometry of submanifolds, focused on discrete nets and their transformations [11, 21, 23]. Some results in this direction were obtained earlier [30]. For instance, the discretization of pseudospherical surfaces is known since more than 50 years [35]. Now it is clear that the transformations of the classical differential geometry (and their discrete analogues) are associated with integrable systems of nonlinear partial differential (and difference) equations and their soliton solutions (see [29, 36]).

In this paper we consider discrete nets, i.e., maps $F : \mathbf{Z}^n \to \mathbf{R}^m$. In the case $n = 2$ they are also called discrete surfaces. The map $\mathbf{R}^n \to \mathbf{R}^m$, obtained in the continuum limit from a discrete net, corresponds to a specific choice of coordinates on some smooth surface.

Some examples are in order. The discrete analogue for asymptotic nets is characterized by the property that any point F and its all four neighbours (T_1F, T_2F, $T_1^{-1}F$, $T_2^{-1}F$) are co-planar. T_j denotes the shift in j-th variable, i.e.,

$$T_j f(m^1, \ldots, m^j, \ldots, m^n) = f(m^1, \ldots, m^j + 1, \ldots, m^n) \ .$$

Discrete pseudospherical surfaces are defined as discrete asymptotic nets such that all segments joining the neighbouring points have equal lengths [10, 35].

By an elementary quadrilateral we mean four neighbouring points: F, T_kF, T_jF and T_kT_jF. Planar quadrilaterals correspond in the smooth case to conjugate nets (i.e., coordinates such that the second fundamental form is diagonal) [22].

Circular nets (such that every quadrilateral is inscribed into a circle) correspond to curvature lines (i.e., coordinates such that both fundamental forms are diagonal) [7, 20].

Isothermic immersions (characterized, in the smooth case, by the property that curvature lines admit conformal (isothermic) parameterization) in the discrete case are defined by the requirement that the cross-ratio for any elemetary quadrilateral is a negative constant [9].

In this paper, following the procedure applied earlier in the smooth case [18], we identify the space $\mathbf{R}^m$ with the vector space V generating the Clifford

algebra $Cl(V)$:

$$\mathbf{R}^m \ni (x^1, \ldots, x^m) \quad \longleftrightarrow \quad x^1\mathbf{e}_1 + \ldots + x^m\mathbf{e}_m \ \in V \subset Cl(V) \ .$$

We recall that the vector space V equipped with a quadratic form of the signature (p,q), $p+q=m$, generates the Clifford algebra $Cl(V) \simeq Cl_{p,q}$. The multiplication in the Clifford algebra $Cl_{p,q}$ satisfies [2, 28]

$$\mathbf{e}_1^2 = \ldots = \mathbf{e}_p^2 = 1 \ , \quad \mathbf{e}_{p+1}^2 = \ldots = \mathbf{e}_{p+q}^2 = -1 \ , \quad \mathbf{e}_j\mathbf{e}_k = -\mathbf{e}_k\mathbf{e}_j \quad (k \neq j) \ ,$$

i.e., for any Clifford vectors $v = v^1\mathbf{e}_1 + \ldots v^m\mathbf{e}_m$ and $w = w^1\mathbf{e}_1 + \ldots + w^m\mathbf{e}_m$ we have

$$vw + wv = 2\langle v \mid w\rangle \equiv 2(v^1w^1 + \ldots v^pw^p - v^{p+1}w^{p+1} - \ldots - v^{p+q}w^{p+q}) \ ,$$

where the right hand side is understood to be proportional to the unit element $\mathbf{1}$ of the algebra $Cl(V)$ (in general, we identify scalars, the one-dimensional linear space spanned by $\mathbf{1}$, with $\mathbf{R}$). In particular, the Clifford square of any vector is real.

The algebra $Cl(V)$ is spanned by $\mathbf{1}$, vectors $\mathbf{e}_k$ and multi-vectors $\mathbf{e}_{k_1}\ldots\mathbf{e}_{k_r}$ ($1 \leq k_1 < k_2 < \ldots < k_r \leq p+q$, $1 < r \leq p+q$) and $\dim Cl_{p,q} = 2^{p+q}$.

The Lipschitz group $\Gamma(V)$ (known also as the Clifford group) is the multiplicative group (with respect to the Clifford product) generated by Clifford vectors. The group generated by unit vectors is called $\mathrm{Pin}(V)$, the group generated by even number of vectors is denoted by $\Gamma_0(V)$, and, finally, the group generated by even number of unit vectors is called $\mathrm{Spin}(V)$ [2, 28]. Obviously,

$$\mathrm{Spin}(V) \subset \mathrm{Pin}(V) \subset \Gamma(V) \subset Cl(V) \ , \quad V \subset \mathrm{Pin}(V) \ , \quad \Gamma_0(V) \subset \Gamma(V) \ .$$

A convenient way to describe circular nets is the **cross ratio** (see, for example, [3]), and especially its generalization for Euclidean spaces ("the Clifford cross ratio") [15]. Namely, for any sequence of 4 points in a Euclidean space we define

$$Q(X_1, X_2, X_3, X_4) := (X_1 - X_2)(X_2 - X_3)^{-1}(X_3 - X_4)(X_4 - X_1)^{-1}. \quad (1)$$

In general $Q(X_1, X_2, X_3, X_4)$ is an element of $\Gamma_0(V)$. In the pseudo-Euclidean case ($pq \neq 0$) there exist non-invertible (isotropic) vectors and, therefore, the cross-ratio is not always well defined. One can easily show the following proposition ([15], compare [9]).

Proposition 1 *The Clifford cross ratio $Q(X_1, X_2, X_3, X_4)$ is real (i.e., proportional to the unit element of $Cl(V)$) if and only if X_1, X_2, X_3, X_4 lie on a circle or are co-linear.*

Therefore the Clifford cross-ratio can be used to characterize discrete analogues of curvature nets, isothermic surfaces etc. The sides of the elementary quadrilateral are given by $D_k F$, $D_j F$, $T_k D_j F$, $T_j D_k F$, where $D_k F := T_k F - F$. We define

$$Q_{kj}(F) := Q(F, T_k F, T_{kj} F, T_j F) = (D_k F)(T_k D_j F)^{-1}(T_j D_k F)(D_j F)^{-1} \ , \tag{2}$$

and formulate the following corollary.

Proposition 2 *The net $F = F(m^1, \ldots, m^n)$ is a circular net if and only if $Q_{kj}(F) \in \mathbf{R}$ for any $k, j \in \{1, \ldots, n\}$.*

An interesting connection between submanifolds (or discrete nets) and integrable systems is provided by the Sym-Tafel formula $F = \Psi^{-1}\Psi,_\lambda$ [13, 32, 33], where Ψ is a solution of some linear problem ("Lax pair") with the spectral parameter λ. This formula was applied in order to discretize pseudospherical and isothermic surfaces [9, 10].

Proposition 3 *We consider the Clifford algebra $Cl(V \oplus W)$, where V, W are vector spaces ($\dim V = q, \dim W = r$) equipped with quadratic forms. Let Ψ is a solution of the following discrete linear problem:*

$$T_j \Psi = U_j \Psi \ , \quad (j = 1, \ldots, n) \tag{3}$$

where $n \leq q$, and $U_j = U_j(m^1 \ldots, m^n, \lambda) \in \Gamma_0(V)$ have the following expansion in the Taylor series around a given λ_0:

$$\begin{aligned} &U_j = U_j^0 + (\lambda - \lambda_0)\ U_j^1 + (\lambda - \lambda_0)^2\ U_j^2 + \ldots \ , \\ &U_j^0 = \mathbf{e}_j B_j \ , \quad U_j^1 = \mathbf{e}_j A_j \ , \qquad A_j \in W, \quad B_j \in V, \quad \mathbf{e}_j \in V \ , \end{aligned} \tag{4}$$

A_j, B_j are assumed to be invertible, and $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are mutually orthogonal unit vectors. We define the discrete net F by the Sym-Tafel formula

$$F = \Psi^{-1}\Psi,_\lambda |_{\lambda=\lambda_0} \ , \tag{5}$$

and, finally, we assume that at a single point $m_0^1, \ldots, m_0^n$ (at least) we have:

$$\Psi(m_0^1, \ldots, m_0^n, \lambda_0) \in \Gamma_0(V) \ , \quad \Psi(m_0^1, \ldots, m_0^n, \lambda) \in \Gamma_0(V \oplus W) \ .$$

Then

- $(F(m^1,\ldots,m^n) - F(m_0^1,\ldots,m_0^n)) \in V \wedge W$
- *F is a circular net if and only if for $k \neq j$*

$$A_k (T_k A_j)^{-1} (T_j A_k) A_j^{-1} \in \mathbf{R} \ . \tag{6}$$

Therefore, F can be always identified with a net in $V \wedge W$.

Proof: The compatibility conditions ($T_k T_j \Psi = T_j T_k \Psi$) for the linear system (3) read

$$(T_k U_j) U_k = (T_j U_k) U_j \ , \tag{7}$$

or, expanding (7) in the Taylor series around $\lambda = \lambda_0$,

$$\begin{aligned} &(T_k U_j^0) U_k^0 = (T_j U_k^0) U_j^0 \ , \\ &(T_k U_j^1) U_k^0 + (T_k U_j^0) U_k^1 = (T_j U_k^1) U_j^0 + (T_j U_k^0) U_j^1 \ , \\ &(T_k U_j^2) U_k^0 + (T_k U_j^1) U_k^1 + (T_k U_j^0) U_k^2 = (T_j U_k^2) U_j^0 + (T_j U_k^1) U_j^1 + (T_j U_k^0) U_j^2 \ , \end{aligned} \tag{8}$$

and so on. We denote $\Psi_0 := \Psi(m^1,\ldots,m^n,\lambda_0)$. Taking into account $U_k(\lambda_0) = U_k^0$, $U_{k,\lambda}(\lambda_0) = U_k^1$ and $T_j \Psi_0 = U_j^0 \Psi_0$, we have

$$\begin{aligned} &D_k F = (T_k \Psi)^{-1} (T_k \Psi)_{,\lambda}|_{\lambda=\lambda_0} - \Psi^{-1} \Psi_{,\lambda}|_{\lambda=\lambda_0} = \Psi_0^{-1} (U_k^0)^{-1} U_k^1 \Psi_0 \ , \\ &T_j D_k F = \Psi_0^{-1} (U_j^0)^{-1} (T_j U_k^0)^{-1} T_j U_k^1 U_j^0 \Psi_0 \ , \\ &(T_k D_j F)^{-1} = \Psi_0^{-1} (U_k^0)^{-1} (T_k U_j^1)^{-1} (T_k U_j^0) U_k^0 \Psi_0 \ , \end{aligned}$$

Applying the first equation of the system (8) we compute

$$(T_k D_j F)^{-1} (T_j D_k F) = \Psi_0^{-1} (U_k^0)^{-1} (T_k U_j^1)^{-1} (T_j U_k^1) U_j^0 \Psi_0 \ ,$$

and, finally,

$$Q_{kj}(F) = \Psi_0^{-1} \left((U_k^0)^{-1} U_k^1 (U_k^0)^{-1} (T_k U_j^1)^{-1} (T_j U_k^1) U_j^0 (U_j^1)^{-1} U_j^0 \right) \Psi_0 \ , \tag{9}$$

To make further simplification we use (4) and take into account that any element of V anti-commutes with any any element of W:

$$Q_{kj}(F) = -\Psi_0^{-1} B_k^{-2} A_k (T_k A_j)^{-1} (T_j A_k) A_j^{-1} B_j^2 \Psi_0 \ .$$

Therefore, using $V \perp W$ and $\Psi_0 \in \Gamma_0(V)$ (because $U_j^0 \in \Gamma_0(V)$), we get

$$Q_{kj}(F) = -B_k^{-2} B_j^2 \ A_k (T_k A_j)^{-1} (T_j A_k) A_j^{-1} \ .$$

which ends the proof of the second statement of the Proposition 3. To prove the first statement we will show that $(T_j F - F) \in V \wedge W$. Indeed,

$$T_j F - F = ((U_j\Psi)^{-1}(U_j\Psi),_\lambda - \Psi^{-1}\Psi,_\lambda)|_{\lambda=\lambda_0} = \Psi_0^{-1}(U_j^0)^{-1}U_j^1\Psi_0 = \Psi_0^{-1}B_j^{-1}A_j\Psi_0 \, .$$

To complete the proof we notice that A_j commutes with any element of $\Gamma_0(V)$, and $\Psi_0^{-1}B_j^{-1}\Psi_0 \in V$. Therefore $\Psi_0^{-1}B_j^{-1}A_j\Psi_0 \in V \wedge W$. □

Proposition 4 *If $U_k^2 = 0$ (in particular, if U_k are linear in λ), then F defined by (3), (4), (5) is a circular net.*

Proof: We are going to show that in this case the condition (6) follows from the compatibility conditions. Indeed, because $U_k^2 = 0$, then from (8) we get $(T_k U_j^1)U_k^1 = (T_j U_k^1)U_j^1$, which can be rewritten as $(T_k A_j)A_k = -(T_j A_k)A_j$. Hence $A_k^{-1}(T_k A_j)^{-1}(T_j A_k)A_j = -1$ and $A_k(T_k A_j)^{-1}(T_j A_k)A_j^{-1} = -A_k^2 A_j^{-2} \in \mathbf{R}$. □

Proposition 5 *If $\dim W = 1$, then F defined by (3), (4), (5) is a circular net in V.*

Proof: If $\dim W = 1$, then the condition (6) is obvious, and $W \wedge V \simeq V$. □

Proposition 6 *If there exists a discrete net $F_A : \mathbf{Z}^n \rightarrow \mathbf{R}^m$ such that $D_k F_A = A_k$, then Proposition 3 can be reformulated as follows: F is a circular net if and only if F_A is a circular net.*

The reduction to the group $\mathrm{Spin}(V \oplus W)$ is always possible as is shown by the following proposition.

Proposition 7 *If B_j and A_j are unit vectors (for $j = 1, \ldots, n$) then $\Psi_0 \equiv \Psi(m^1, \ldots, m^n, \lambda_0) \in \mathrm{Spin}(V)$ and F takes values in $\mathrm{Spin}(V \oplus W)$.*

Actually, simple bivectors of the form $\hat{v} \wedge \hat{w}$ (where $\hat{v}$, $\hat{w}$ are unit vectors from V) belongs both to $\mathrm{Spin}(V)$ and to the Lie algebra of $\mathrm{Spin}(V)$. Therefore, F takes values also in the Lie algebra of $\mathrm{Spin}(V \oplus W)$ provided that the assumptions of Proposition 7 are satisfied.

We recall the so called main anti-automorphism β of the Clifford algebra (known also as the reversion) [2, 28]:

$$\beta(\mathbf{v}_1\mathbf{v}_2\ldots\mathbf{v}_k) := \mathbf{v}_k\mathbf{v}_{k-1}\ldots\mathbf{v}_1 \ ,$$

$$\beta(c_1X + c_2Y) = c_1\beta(X) + c_2\beta(Y) \ ,$$

for any $\mathbf{v}_j \in V$, any $X, Y \in Cl(V)$, and $c_1, c_2 \in \mathbf{R}$. The group Spin(V) consists of products of unit vectors which means that $X \in$ Spin(V) if and only if $\beta(X)X = \pm 1$.

The $\Gamma(V \oplus W)$-valued spectral problem given by (3), (4) can always be transformed into Spin$(V \oplus W)$-valued spectral problem $T_k\Phi = \hat{U}_k\Phi$ by the transformation $\Phi := g\Psi$, where $g := |\beta(\Psi)\Psi|^{-1/2}$ is a real function, and $\hat{U}_k := g^{-1}(T_kg)U_k$.

Let $P : W \to \mathbf{R}$ is a projection (linear homomorphism of vector spaces satisfying $P^2 = P$). We extend its action on $V \wedge W$ in a natural way. Namely, if $\mathbf{v}_k \in V$ and $\mathbf{w}_k \in W$, then

$$P(\sum_k \mathbf{v}_k\mathbf{w}_k) := \sum_k P(\mathbf{w}_k)\mathbf{v}_k \ .$$

Proposition 8 *Let P is a projection and F is defined by (3), (4), (5). Then $P(F)$ is a circular net.*

Proof: We denote $P(A_k) = a_k \in \mathbf{R}$. To compute $Q_{kj}(P(F))$ we need:

$$D_kP(F) = P(D_kF) = a_k\Psi_0B_k^{-1}\Psi_0 \ ,$$

$$(T_kD_jP(F))^{-1} = (T_ka_j)^{-1}\Psi_0^{-1}B_k^{-1}\mathbf{e}_k^{-1}(T_kB_j)\mathbf{e}_kB_k\Psi_0 \ ,$$

$$T_j(D_k(F)) = (T_ja_k)\Psi_0^{-1}B_j^{-1}\mathbf{e}_j^{-1}(T_jB_k)^{-1}\mathbf{e}_jB_j\Psi_0 \ ,$$

$$(D_jP(F))^{-1} = a_j^{-1}\Psi_0^{-1}B_j\Psi_0 \ .$$

The compatibility conditions $(T_kU_j^0)U_k^0 = (T_jU_k^0)U_j^0$, after taking into account $\mathbf{e}_j\mathbf{e}_k^{-1} = -\mathbf{e}_k^{-1}\mathbf{e}_j$, are equivalent to $\mathbf{e}_k^{-1}(T_kB_j)\mathbf{e}_kB_k = -\mathbf{e}_j^{-1}(T_jB_k)\mathbf{e}_jB_j$. Therefore

$$Q_{kj}(P(F)) = -\frac{a_k(T_ja_k)}{a_j(T_ka_j)}\Psi_0^{-1}B_k^{-1}B_k^{-1}B_jB_j\Psi_0 = -\frac{a_k(T_ja_k)B_j^2}{a_j(T_ka_j)B_k^2} \in \mathbf{R} \ ,$$

which completes the proof (compare [15]). $\square$

We proceed to several examples, where U_j are rational with respect to λ (usually even linear in λ). All assumptions of Proposition 3 are assumed to be satisfied. We denote by $\mathbf{e}_1, \ldots, \mathbf{e}_q$ and $\mathbf{e}_{q+1}, \ldots, \mathbf{e}_{q+r}$ orthonormal bases in V and W, respectively.

The corresponding smooth (continuum) cases were considered in [17, 18, 19], and also (by different approaches) in [1, 9, 12, 26, 27, 31] and earlier [24, 25].

If U_j are linear in λ, then it is not difficult to construct the Darboux-Bäcklund transformation (similarly as in the smooth case, [6]). The details, analogical to the special case of discrete isothermic surfaces [16], will be presented elsewhere.

Discrete isothermic surfaces in $\mathbf{R}^q$

$$V \simeq \mathbf{R}^q \ , \quad W \simeq \mathbf{R}^{1,1} \ , \quad n = 2 \ , \quad r = 2 \ , \quad U_j = U_j^0 + \lambda U_j^1 \ ,$$
$$P(\mathbf{e}_{q+1}) = 1 \ , \quad P(\mathbf{e}_{q+2}) = \pm 1 \ .$$

Smooth isothermic immersions admit isothermic (isometric) parameterization of curvature lines. In these coordinates $ds^2 = \Lambda((dx^1)^2 + (dx^2)^2)$ and the second fundamental form is diagonal.

Discrete Guichard nets in $\mathbf{R}^q$

$$V \simeq \mathbf{R}^q \ , \quad W \simeq \mathbf{R}^{2,1} \ , \quad n = 3 \ , \quad r = 3 \ , \quad U_j = U_j^0 + \lambda U_j^1 \ ,$$
$$P(\mathbf{e}_{q+1}) = \cos\varphi_0 \ , \quad P(\mathbf{e}_{q+2}) = \sin\varphi_0 \ , \quad P(\mathbf{e}_{q+3}) = \pm 1 \ , \quad \varphi_0 = \mathrm{const} \ .$$

Guichard nets in $\mathbf{R}^3$ are characterized by the constraint $H_1^2 + H_2^2 = H_3^2$, where H_j are Lamé coefficients, i.e., $ds^2 = H_1^2(dx^1)^2 + H_2^2(dx^2)^2 + H_3^2(dx^3)^2$.

Discretization of some class of orthogonal nets in $\mathbf{R}^n$

$$V \simeq \mathbf{R}^n \ , \quad W \simeq \mathbf{R}^n \ , \quad q = n \ , \quad r = n \ , \quad U_j = U_j^0 + \lambda U_j^1 \ ,$$
$$P(\mathbf{e}_{n+k}) = 1 \ , \quad P(\mathbf{e}_{n+j}) = 0 \quad (j \neq k) \ .$$

This class in the smooth case is defined by the constraint $H_1^2 + \ldots + H_n^2 =$ const, where H_j are Lamé coefficients, i.e., $ds^2 = H_1^2(dx^1)^2 + \ldots + H_n^2(dx^n)^2$.

Discrete Lobachevsky n-spaces in $\mathbf{R}^{2n-1}$

$$V = V_1 \oplus V_2 \ , \quad V_1 \simeq \mathbf{R}^n \ , \quad V_2 \simeq \mathbf{R}^{n-1} \ , \quad W \simeq \mathbf{R} \ , \quad \lambda_0 = 1 \ ,$$

$$U_j = \mathbf{e}_j \left(\tfrac{1}{2} \left(\lambda - \tfrac{1}{\lambda} \right) A_j + \tfrac{1}{2} \left(\lambda + \tfrac{1}{\lambda} \right) P_j + Q_j \right) \ ,$$

$$\mathbf{e}_j \in V_1 \ , \quad Q_j \in V_1 \ , \quad P_j \in V_2 \ , \quad A_j \in W \ , \quad P_j + Q_j = B_j \ .$$

In the continuum limit we get immersions with the constant negative sectional curvature (Lobachevsky spaces) [4, 5, 34]. The discrete case is presented in more detail in [14].

In all presented cases the continuum limit, done by assuming $\varepsilon \to 0$ where ε is the size of the $\mathbf{Z}^n$ lattice (compare [8, 9]), seems to be correct because all algebraic properties (including the integrability) are preserved by our discretization.

However, a purely geometrical characterization is known only in the case of isothermic surfaces: the cross ratio is harmonic [9]. Therefore a geometrical characterization of discrete nets presented in my paper is an important open problem. The general characterization of families of smooth submanifolds corresponding to the discrete nets described in Proposition 3 (see also [19]) is also not done.

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